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RESEARCH METHODS AND STATISTICS

An Emergency Department Patient Flow Model Based on Queueing Theory Principles

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Abstract

Objectives: The objective was to derive and validate a novel queuing theory–based model that predicts the effect of various patient crowding scenarios on patient left without being seen (LWBS) rates.

Methods: Retrospective data were collected from all patient presentations to triage at an urban, academic, adult-only emergency department (ED) with 87,705 visits in calendar year 2008. Data from specific time windows during the day were divided into derivation and validation sets based on odd or even days. Patient records with incomplete time data were excluded. With an established call center queueing model, input variables were modified to adapt this model to the ED setting, while satisfying the underlying assumptions of queueing theory. The primary aim was the derivation and validation of an ED flow model. Chi-square and Student's t-tests were used for model derivation and validation. The secondary aim was estimating the effect of varying ED patient arrival and boarding scenarios on LWBS rates using this model.

Results: The assumption of stationarity of the model was validated for three time periods (peak arrival rate = 10:00 a.m. to 12:00 p.m.; a moderate arrival rate = 8:00 a.m. to 10:00 a.m.; and lowest arrival rate = 4:00 a.m. to 6:00 a.m.) and for different days of the week and month. Between 10:00 a.m. and 12:00 p.m., defined as the primary study period representing peak arrivals, 3.9% (n = 4,038) of patients LWBS. Using the derived model, the predicted LWBS rate was 4%. LWBS rates increased as the rate of ED patient arrivals, treatment times, and ED boarding times increased. A 10% increase in hourly ED patient arrivals from the observed average arrival rate increased the predicted LWBS rate to 10.8%; a 10% decrease in hourly ED patient arrivals from the observed average arrival rate predicted a 1.6% LWBS rate. A 30-minute decrease in treatment time from the observed average treatment time predicted a 1.4% LWBS. A 1% increase in patient arrivals has the same effect on LWBS rates as a 1% increase in treatment time. Reducing boarding times by 10% is expected to reduce LWBS rates by approximately 0.8%.

Conclusions: This novel queuing theory–based model predicts the effect of patient arrivals, treatment time, and ED boarding on the rate of patients who LWBS at one institution. More studies are needed to validate this model across other institutions.

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Emergency department (ED) crowding has received considerable national attention in recent years.[1,2] From 1996 to 2006, the annual number of ED visits in the United States increased nearly 32%, from 90.3 million to 119.2 million, while the number of hospital EDs decreased nearly 5%.[3] The holding in the ED of patients admitted to the hospital (ED boarding) has also been noted to be a growing problem and is a large contributor to ED crowding.[4] ED crowding is known to increase patient wait times.[4–6] From 1997 to 2004 and then to 2006, the median wait time to see a physician increased from 38 to 47 to 56 minutes, an increase of 36%.[7] As wait times increase, the rate of patients who leave without being seen

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(LWBS) also increases.[8–14] ED patients who leave with being evaluated by a physician are at risk for poorer health outcomes,[10,13,15] represent a source of lost revenue for hospitals,[16] and decrease patient satisfaction.[10] For these and other reasons, research efforts have been directed at predicting ED patient load volumes to inform real-time operational interventions with the objectives of managing surge conditions, crowding, and wait times. Various approaches to forecast ED patient volumes have been proposed and studied.[17–22] However, no approach has been demonstrated to reliably define crowded conditions when applied to diverse practice settings[23] or to reliably outperform simple indices such as bed occupancy rate.[24] Some have proposed queueing theory as a logical next step in modeling ED census and crowding.[2,25,26]

Queueing theory makes basic assumptions about a system to create mathematical equations that describe system flow when there are variable inputs and fixed resources. Derived in large part from the telecommunications industry, this methodology has a potentially useful application to the ED setting, where patient flow modeling could predict patient waits. Although many service industry–related queuing models exist, application to the ED setting has been limited.[27–37] Previous work has considered portable radiology workflow,[28] meeting specific ED disposition time targets,[31] ED staffing,[34] hospital bed resource allocation, priorities for admission,[35] revenue losses from LWBS patients,[36] fast track,[37] and prehospital operations.[38–41] None has been used to predict patients who LWBS based on wait time tolerance and ED crowding.

Therefore, our primary aim was the derivation and validation of an ED flow model based on the novel modification of a queueing model commonly used in the call center industry. The secondary aim was estimating the effect of varying ED patient arrival and boarding scenarios on LWBS rates using this model.

METHODS

Study Design

The established queueing model $M/GI/r/s + GI$[42] describes customer reneging (leaving system before completing evaluation) after prolonged call center wait times (see Table 1 for explanation of terms). It is the most accurate available queueing model that describes highly variable systems where multiple customers (patients) are served (treated) in parallel, while allowing customer reneging, multiple servers, and a finite waiting room volume. Using this established call center queueing model, we modified model input variables to adapt this model to the ED setting (Table 1), while satisfying the underlying assumptions of queueing theory. This study was approved by the institutional review board.

Study Setting and Population

Data for all patients who registered at triage during calendar year 2008 were collected from an urban academic adult-only ED with an annual volume of 87,705 patients, using the institution's electronic medical record system (Healthmatics ED, Allscripts, Chicago, IL). Visits with missing or incomplete operational patient flow metric data were excluded ($n = 647$). Time stamps for operational patient flow metrics obtained were: 1) patient arrival time; 2) Emergency Severity Index triage acuity score; 3) ED bed placement time; 4) patient time to LWBS (this time stamp occurs when patient is called from waiting area to be placed in ED patient treatment area, but does not respond; most who leave do not notify ED staff; the institution's practice is to call the patient three different times, and record a time stamp for each attempt, and the patient is recorded as a LWBS after third attempt; the first of these attempt time stamps was used for our analysis, as an approximation of the LWBS time); 5) total treatment time (time interval from patient sign-in at triage to time patient is admitted, discharged, transferred to another facility, leaves against medical advice, or expires); and 6) ED boarding time (defined as starting 2 hours after decision to admit was documented).

Study Protocol

A complete description of the model derivation and specification are provided in Data Supplement S1 (available as supporting information in the online version of this paper). Briefly, as in the article by Whitt,[42] we approximate the $M/GI/r/s + GI$ model with the

Table 1
Summary of Queueing Model $M/M/r/s + M(n)$ Inputs

Queueing Model Term	Call Center Application	Modification for ED System
First M	Interarrival times between calls to the system assumed to follow an exponential distribution.*	Time between ED patient arrivals.
Second M	Time speaking to call center agent follows an exponential time distribution.	Treatment time (including boarding).
r	Number of agents available to take calls.	Total ED treatment space (bed) capacity.
s	Maximum capacity of call center to accommodate calls.	Waiting area capacity (i.e., maximum number of patients who will wait for evaluation).
$M(n)$	Caller waiting time tolerance distribution approximated by an exponential distribution as a function of total number of callers waiting.	Patient waiting time tolerance to see provider calculated from a Weibull distribution.46

*Arrivals occur with a known average rate and the number of arrivals in some fixed time period are independent of the number of arrivals in a nonoverlapping time period.[20]

established queueing model $M/M/r/s + M(n)$ with inputs adapted to describe ED patient flow (Table 1). Assumptions required to derive and specify the $M/M/r/s + M(n)$ model are described in Table 2. During the primary study period, arrivals between 10:00 a.m. and 12:00 p.m. were adequately stationary (i.e., stable patient flow) and represented 12.3% of ED patient daily arrivals. We have chosen this time period as our key period of study because it has the largest arrival rate of the day for our studied institution, as is demonstrated in Supplement Figure 1. To evaluate stationarity of the arrival rate during the study period we use a method similar to that of Brown et al.[43] who tested stationarity of arrivals in their call center model. We tested for stationarity of arrivals between 10:00 a.m. to 12:00 p.m., and at a 1% significance level, we found no reason to reject the stationarity assumption in 96% of the observed days.

In addition to these observations for stationarity during each day, we have tested stationarity of arrivals among different days. To do this, we divided all days into 12 bins (each bin represents a month) to estimate standard deviation of arrival rate in this period of time. We used a Student's t distribution with 11 degrees of freedom to find 95% confidence intervals (CIs) for arrival rates, and we found that all arrival rates lie within this 95% CI. Therefore, there is no reason to reject stationarity of arrival rates among different months of the year. To further test stationarity, we also divided our dataset into seven bins (each bin represents a day of a week).

To validate our model we selected the 2-hour period with a moderate patient arrival rate (8:00 a.m. to 10:00 a.m.) and the 2-hour period with the lowest arrival rate (4:00 a.m. to 6:00 a.m.). Given these observations, we

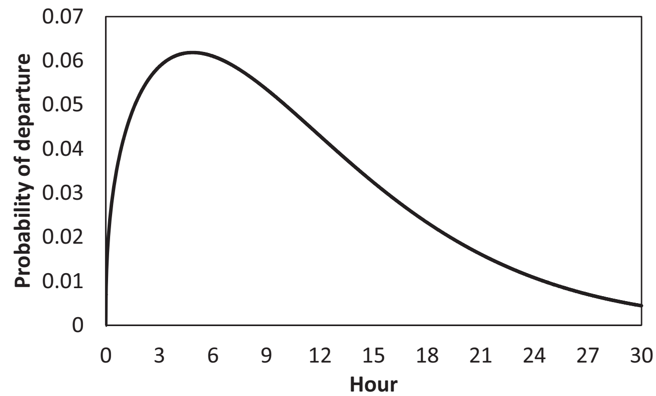

Figure 1. Weibull distribution of patients' estimated waiting time tolerance.

believe that our defined study period is quite stationary at the daily level within the prescribed time windows; therefore, we can apply our queueing tools to analyze patients' LWBS behavior in this period. However, as queueing models are heavily reliant on stationarity assumptions, readers are cautioned both to test for stationarity before applying the models of this article in their settings and to realize that the results apply narrowly for the modeled stationary period.

The number of ED treatment spaces at the study institution varied depending on the time of day, because the fast track area was closed in the late evenings, and a 12-bed observation unit with one additional provider could be used for ED boarding patients as a surge overflow for additional bed capacity when needed. To

Table 2
Required Assumptions and Inputs of a Queueing Theory Modeling

Required Assumptions and Inputs	Description	Methodology	Application to Study ED
Stationarity	Stable flow (i.e., rates of arrivals and departures are constant).	Models typically use a relatively stationary period for service demand (e.g., lunchtime for fast food queueing analysis).	Arrivals between 10:00 a.m. and 12:00 p.m. were adequately stationary for analysis.
Interarrival and service probability distribution	Rate of arrivals and service delivery.	Continuous standard distribution models (e.g., lognormal, logistic, Student's t, Weibull, beta, etc) are tested to determine the best fit.	Daily patient interarrival and ED length of stay exponential distributions were approximated using standard maximum likelihood estimation methodology.47
Prioritization	Order in which customers are serviced.	Many (e.g., first-come first-serve, first-come first-out, last-in first-out)	Requires collapse of patient acuity segmentation (e.g., emergency severity index classification) into a single acuity class in order to be mathematically tenable.
Server	Fixed capacity to service customers.	Servers can be providers (e.g., bank teller) or space (e.g., number of ED beds).	Because fast track and observation areas were used as needed in times of crowding, a calculated "effective number of beds" was used.
Queue capacity	Describes how long the line to receive services is or can be.	Once this "capacity" is saturated, all patients are diverted out of system.	Waiting area capacity expanded to accommodate walk-in patients but finite capacity set to model ambulance diversion.
Waiting time tolerance	Assumed each customer has a wait time tolerance that is independent of others waiting in the queue.	A general distribution is allowed for tolerance.	The Weibull distribution was applied to describe actual tolerance.

account for this, a fixed bed capacity r was approximated for the model and designated as the *effective number of beds*. Thus, the number of "effective beds" was defined as a fixed bed capacity using data from odd days for the time period of 10:00 a.m. to 12:00 p.m. (the same key time period used for the stationarity assumption). The term "s" describes the total number of patients who will wait for evaluation. It was assumed that once this "capacity" is saturated, all patients are diverted out of the system. Because this does not account for walk-in patients, the modeled waiting capacity was expanded to appropriately describe the study ED walk-in volume. This is consistent with previous studies that have also modeled ambulance diversion and excessive patient waits using a fixed waiting area capacity estimate.[44,45]

Data Analysis

We used a Weibull probability distribution[46] to describe patient wait time tolerance (Figure 1). To predict LWBS rates for the system, we modified the call center $M/GI/r/s + GI$ queueing model using a methodology described by Whitt.[42] Whitt developed an algorithm to rapidly compute approximations for all of the standard steady-state performance measures in the basic call center queueing model $M/GI/s/r + GI$, which has a Poisson arrival process, independent and identically distributed service times with a general distribution, s servers, r extra waiting spaces, and customer abandonment times with a general distribution. Simulation experiments by Whitt showed that the approximation is quite accurate to predict abandonment in call center customers.[42] In this article, we have applied this algorithm to determine the LWBS rate from our studied ED after finding and calibrating the best fit distributions for interarrival, treatment, and tolerance times. We used maximum likelihood estimation[47] and the expectation maximization[48] algorithms to identify distributional parameters of waiting time tolerance and total "ED bed occupancy" (sum of ED treatment and boarding time) to fully specify the model.

Data were divided into derivation (odd dates) and validation (even dates) sets. We divided the derivation data set into 15 time-based bins spread across time (pattern of bin 1 = days 1, 31, 61, 91, …; bin 2 = days 3, 33, 63, 93, …; bin 15 = 29, 59, 89, 119, …, etc. repeated through 365 days) and calculated the LWBS rate for each bin. We used the derivation set to find the standard deviation of the observed LWBS rate. We then used the estimated SD to find a 95% CI for the predicted LWBS rate of the validation set based on a Student's t-test with 14 degrees of freedom, which is 95% CI = 0.41 to 6.92. To validate our model, we used the validation data set to compute input parameters of our model including average arrival rate and service and boarding times. Substituting for these parameters in our model we computed predicted LWBS rate for the validation data set. Because the predicted LWBS (2.75%) lies in the computed CI, we did not find any evidence to reject validity of our model. We also performed a secondary validation using data from weekends, determined a priori to be a more homogeneous time period. Similar to the previous validation test we computed a 95% CI for the predicted LWBS rate of the validation set based on a Student's t-test with 14 degrees of freedom: 95% CI = 0 to 4.59 (which is quite different from the previous validation test since weekends are a more homogeneous period). Again, our predicted LWBS rate (3%) from our model lies in this CI, which provides no evidence to reject validity of our model. We determined the effect of ED patient arrival rates, treatment times, and boarding on LWBS rates using the validated model.

RESULTS

To validate our model we selected a 2-hour period with a moderate patient arrival rate (8:00 a.m. to 10:00 a.m., total arrival of 8,304 patients, 9.5% of all ED visits) and a 2-hour period with the lowest arrival rate (4:00 a.m. to 6:00 a.m., total arrival of 4,239 patients, 4.8% of all ED visits). As we mentioned earlier, the test described by Brown et al.[43] demonstrated that arrivals in these periods were stationary, so we could apply our model to predict LWBS rate in these periods. Using the same methodology for each of these periods, we found that for the time period of 8:00 a.m. to 10:00 a.m., the 95% CI for LWBS rate on even days is 95% CI = 0 to 5.25. Since our model prediction is 1%, which lies in this interval, there is no evidence to reject our model prediction. Similarly for the period of 4:00 a.m. to 6:00 a.m., the CI for the LWBS rate on even days is 95% CI = 0 to 5.65. Since our model prediction (2%) lies within this interval, again there is no evidence to reject our model prediction.

We also performed a secondary validation test using data from weekends. We used arrivals on Saturday as the derivation set to calibrate our model and compute 95% CI for the predicted LWBS rate of the validation set (which is arrivals on Sunday) based on a Student's t-test with 14 degrees of freedom. For the period of 8:00 a.m. -10:00 a.m., the 95% CI for LWBS rate on Sundays is 0 to 4.25. Since our model prediction is 2%, there is no evidence to reject our model prediction. Similarly, for the period of 4:00 a.m. to 6:00 a.m., the LWBS rate on even days is 95% CI = 0.15 to 6.65, which includes our model prediction of LWBS rate on Sundays, so there is no evidence to reject our model prediction.

Estimating the Effect of ED Patient Arrivals and Boarding on LWBS Rates Using a Novel Model

The effect of varying ED patient arrivals and boarding on LWBS rates was determined using the key stationary time period, defined as being between 10:00 a.m. and 12:00 p.m. (i.e., peak patient arrival time; see Table 2 for definition), when 4.1% of patients (n = 418) LWBS by a provider. The mean (±SD) wait time tolerance for the system (i.e., actual study ED population) during this time period was 10.68 (±7.76) hours. The actual versus model-predicted LWBS rates are presented in Table 3. The average wait time to see a provider using the model was 85 minutes, which was very close to the actual average wait time of 89 minutes.

The effect of ED arrivals by hour on LWBS rates is provided in Figure 2. A 10% increase in hourly ED patient arrivals from the observed average arrival rate predicted a 10.74% LWBS rate. A 10% decrease in

Table 3
Actual Versus Model Predicted LWBS Rates for Key Study Period (10:00 a.m. to 12:00 p.m.)

Quarter	Observed LWBS	Modeled LWBS	Lower 95% CI Interval	Upper 95% CI interval
1	6.7	9	3.7	9.7
2	3.3	2.1	0.3	6.3
3	3.5	4.8	0.5	6.5
4	2.6	2.9	0	5.6

Data are reported as percentages.
LWBS = left without being seen.

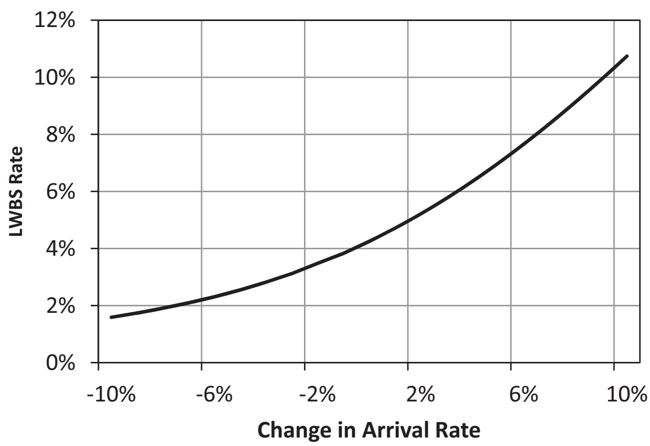

Figure 2. Effect of ED arrivals by hour on LWBS rates. LWBS = left without being seen.

hourly arrivals from the observed average arrival rate predicted a 1.6% LWBS rate. The duration of treatment time also influences the rate of LWBS (Figure 3). Specifically, a 30-minute decrease in treatment time from the observed average treatment time predicts a 1.4% LWBS rate. It was observed that a 1% increase in the rate of ED patient arrivals has the same effect on LWBS rates as a 1% increase in treatment time.

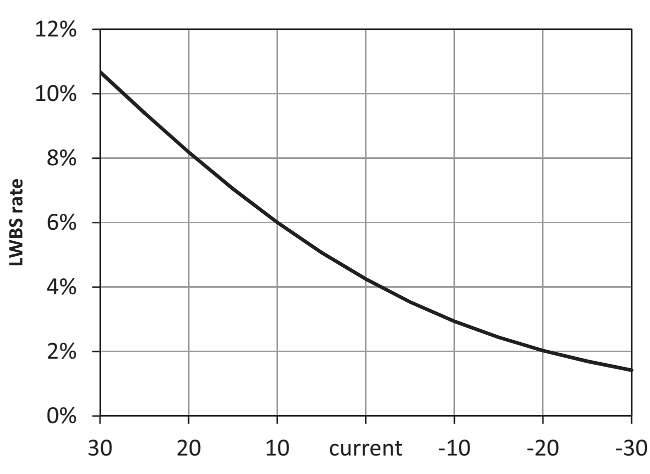

Figure 3. Duration of treatment time influences LWBS rates. LWBS = left without being seen.

Reducing the number of admitted patients in the ED who are waiting for an inpatient bed i.e., "boarding", notably reduces LWBS rates (Figure 4). Reducing boarding times by 10% is expected to reduce LWBS rates by approximately 0.8%, with a 50% reduction expected to decrease the LWBS rate to 1.5% (from the actual 4%) in the study ED.

DISCUSSION

Crowding is known to prompt patients to leave EDs without being seen by providers. Numerous studies have catalogued the characteristics of patients who LWBS, but none has described a mathematical prediction tool to help inform ED operations.[8–15] Queueing models lend themselves well to describing the ED environment because they allow for the application of simple equations to model patient flow. In most ED applications these equations can be easily input into a spreadsheet. At the most basic level, queueing systems consist of four components: arrivals, servers, service principles (described as the "queueing discipline" or rules as to whom a server serves next), and the flow or routing of the customer or item through the system. These models then describe the effect of varying demand on wait times, waiting tolerance, capacity, and utilization metrics. A handful of queueing models have been designed to describe ED patient flow[29,31] and predict demand in the ED.[27,30] However, none has been ideal to describe the complex ED environment, nor describe the effect of patient demand on LWBS rates.

Using our calibrated $M/M/r/s + M(n)$ model (Table 1), we found that ED LWBS rates climb in a predictable and exponential way as the rate of ED patient arrivals increases. This is not surprising, since as more patients arrive per hour, the queue to be served (i.e., bed placement in our model) grows, resulting in longer wait times for patients. Strategies that obviate the need for bed placement (e.g., treat and release by a provider in triage) would be expected to have a positive effect on LWBS rates, but these were not modeled in our study.

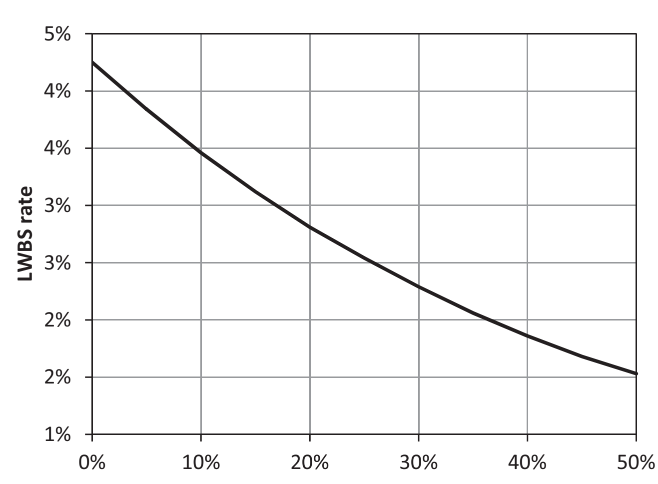

Figure 4. Reducing the number of admitted patients in the ED who are waiting for a inpatient beds ("boarding") reduces LWBS rates. LWBS = left without being seen.

The overall mean waiting time tolerance in our patient population was nearly 11 hours, and indeed most patients (93% of all arrivals to ED) stayed for treatment rather than LWBS. Identifying the patient wait time tolerance distribution(s) was challenging because the study data necessarily censored for those who remained in queue and were ultimately evaluated by a physician. Our ED is located in an urban center and had notable issues with crowding during the study period. It is not known if the waiting time tolerance of patients at the study institution were affected by the regular crowded conditions or if the wait time tolerance mirrors that of other ED populations. However, this is the first description of patient waiting time tolerance in the ED setting to our knowledge.

What was surprising is the dramatic effect that improved treatment (service) time is expected to have on patients that LWBS. Our model predicts a decrease of current LWBS rates from over 3.9% to 1.4% with a reduction of 30 minutes in average service time. While longer ED lengths of stay are known to increase LWBS rates[8–14] and decrease patient satisfaction,[10] improvements in length of stay trends may have dramatic effects on those who would otherwise LWBS.

Reducing ED patient boarding directly improves LWBS rates according to our model. This is consistent with other studies that have found that ED boarding is a significant contributor to ED crowding.[4,49] The demonstration of the impressive effects that ED boarding has on LWBS rates may be valuable to ED and hospital administrators as they attempt to prioritize ED operations change management strategies to minimize patients who LWBS. It has been estimated that each patient who LWBS represents $858 of lost charge revenue to the institution.[16] Tools such as our model, which help to predict the effect on LWBS rates of various interventions on arrival rates, treatment, and boarding time, can be helpful not only to improve access to care for emergency patients and to shield the institution from possible medicolegal risk exposure, but also to capture lost revenue.

LIMITATIONS

Although no system is directly analogous to the ED, call centers are in many ways similar to ED systems. They are nonstationary (i.e., nonconstant flow), have "triage" in the form of the automated voice system that usually guides one to an agent, and are complex environments (with sizable caller and staff heterogeneity). Queueing models have been shown to appropriately model the call center environment,[42] despite the assumptions that the model(s) require (see Tables 1 and 2).

Queueing models require assumptions that 1) customer flow is unidirectional (patients move through the system from one service to the next (queue) with no unscheduled delays for resources other than the one server they are queueing for), 2) arrivals are unpredictable (or predictable but unmanageable), and 3) the arrival rate of the system is stationary and constant over time (i.e., stationarity). Patient arrivals of the modeled system were relatively stationary on our study periods from 4:00 a.m. to 6:00 a.m., 8:00 a.m. to 10:00 a.m., and 10:00 a.m. to 12:00 p.m. The final period (10:00 a.m. to 12:00 p.m.) has the largest arrival rate of the day for our studied institution, as is demonstrated in Supplement Figure 1, which shows total number of arrivals to triage during a day and was the key period studied. Altogether, we find that in any stationary period of time, our model provides a reasonably good estimate of LWBS rates, which demonstrates the clinical utility of our method. We note that our model may not be used if arrival to an ED is not relatively stationary; if applied to another institution, our model could potentially result in significant errors in its predictions.

A standard queueing model has an infinite capacity and assumes that people will wait indefinitely, even if there are thousands of people queued before them. We felt that such a model was less realistic than one with a finite capacity. Our model requires a fixed queue capacity (s waiting room capacity), which was expanded to describe the average walk-in traffic patterns of the study ED. This model, like other queueing models, does not accommodate a "no diversion policy." This is consistent with previous studies, which have also modeled ambulance diversion and excessive patient waits using a fixed waiting area capacity estimate.[44,45]

At times the study ED accommodated boarding patients in the observation unit. The reality is that EDs often have to expand their capacity, e.g., using hallway beds, to care for patients during times of demand surge. These practice variables based on capacity needs do not lend themselves well to modeling. Therefore, we calculated an estimated number of beds (i.e., "effective number of beds"), which correlated well with our service metrics (see Data Supplement S1).

Unfortunately, no previously reported queueing model is ideal for the patient acuity prioritization of ED patients. Currently a few queueing models describe either multiclass (i.e., multiacuity) customers or reneging from a system,[50,51] but none adequately describe both. We attempted to address these issues without creating a mathematically complex and untenable model. Because our primary aim was to specify a model that predicted LWBS rates, acuity had to be collapsed.

At the time of this study, boarding time was defined as beginning 2 hours after the admission order was placed. This has been an area of national debate, with the latest consensus now defining boarding time as beginning contemporaneously with placement of the admission order. Use of this most recent definition would add 2 hours to our individual boarding times and would affect our results accordingly.

We believe that our defined study period is quite stationary at the daily level within the prescribed time windows, and therefore we can apply our queueing tools to analyze patients' LWBS behavior in this period. However, as queueing models are heavily reliant on stationarity assumptions, readers are cautioned both to test for stationarity before applying the models of this article in their setting and to realize that the results apply narrowly for the modeled stationary period.

This model was derived and validated using data from a single center and might have limited generalizability. However, despite the stated assumptions of our model, we believe that it is a logical next step for the queueing models we already have, but note that additional work is needed to create mathematically stable models that can take the realities of emergency practice into account.

CONCLUSIONS

Creating mathematical models that adequately describe the unique and complex flow of patients in the ED is challenging. We present a novel queueing theory that predicts patient wait times and quantifies the effect of ED patient arrivals, treatment times, and boarding on leave without being seen rates. Future studies are needed to validate this model across various institutions.

References

1. Moskop JC, Sklar DP, Geiderman JM, et al. Emergency department crowding, part 1: concept, causes, and moral consequences. Ann Emerg Med. 2009; 53:605–11.
2. Hoot NR, Aronsky D. Systematic review of emergency department crowding: causes, effects, and solutions. Ann Emerg Med. 2008; 52:126–36.
3. Pitts SR, Niska RW, Xu J, et al. National Hospital Ambulatory Medical Care Survey: 2006 emergency department summary. Natl Health Stat Rep. 2008; 7:1–38.
4. United States Government Accountability Office. Hospital Emergency Departments; Crowding Continues to Occur, and Some Patients Wait Longer Than Recommended Time Frames. Available at: http://www.gao.gov/new.items/d09347.pdf. Accessed Jul 9, 2013.
5. Horwitz LI, Green J, Bradley EH. US emergency department performance on wait time and length of visit. Ann Emerg Med. 2010; 55:133–41.
6. Wilper AP, Woolhandler S, Lasser KE, et al. Waits to see an emergency department physician: US trends and predictors, 1997–2004. Health Aff (Millwood). 2008; 27:w84–95.
7. Pitts SR, Niska RW, Xu J, et al. National Hospital Ambulatory Medical Care Survey: 2006 emergency department summary. Natl Health Stat Rep. 2008; 6:1–38.
8. Ding R, McCarthy ML, Li G, et al. Patients who leave without being seen: their characteristics and history of emergency department use. Ann Emerg Med. 2006; 48:686–93.
9. Asaro PV, Lewis LM, Boxerman SB. Emergency department overcrowding: analysis of the factors of renege rate. Acad Emerg Med. 2007; 14:157–62.
10. Kennedy M, MacBean CE, Brand C, et al. Review article: leaving the emergency department without being seen. Emerg Med Australas. 2008; 20:306–13.
11. Mohsin M, Forero R, Ieraci S, et al. A population follow-up study of patients who left an emergency department without being seen by a medical officer. Emerg Med J. 2007; 24:175–9.
12. Goodacre S, Webster A. Who waits longest in the emergency department and who leaves without being seen? Emerg Med J. 2005; 22:93–6.
13. Rowe BH, Channan P, Bullard M, et al. Characteristics of patients who leave emergency departments without being seen. Emerg Med. 2006; 13:848–52.
14. Polevoi SK, Quinn JV, Kramer NR. Factors associated with patients who leave without being seen. Acad Emerg Med. 2005; 12:232–6.
15. Baker DW, Stevens CD, Brook RH. Patients who leave a public hospital emergency department without being seen by a physician. Causes and consequences. JAMA. 1991; 266:1085–90.
16. Falvo T, Grove L, Stachura R, et al. The financial impact of ambulance diversions and patient elopements. Acad Emerg Med. 2007; 14:58–62.
17. Flottemesch TJ, Gordon BD, Jones SS. Advanced statistics: developing a formal model of emergency department census and defining operational efficiency. Acad Emerg Med. 2007; 14:799–809.
18. Hoot NR, Epstein SK, Allen TL, et al. Forecasting emergency department crowding: an external, multicenter evaluation. Ann Emerg Med. 2009; 54:514–22.
19. Jones SS, Thomas A, Evans RS, et al. Forecasting daily patient volumes in the emergency department. Acad Emerg Med. 2008; 15:159–7.
20. McCarthy ML, Zeger SL, Ding R, et al. The challenge of predicting demand for emergency department services. Acad Emerg Med. 2008; 15:337–46.
21. Hoot NR, Zhou C, Jones I, et al. Measuring and forecasting emergency department crowding in real time. Ann Emerg Med. 2007; 49:747–55.
22. Hoot NR, LeBlanc LJ, Jones I, et al. Forecasting emergency department crowding: a discrete event simulation. Ann Emerg Med. 2008; 52:116–25.
23. Jones SS, Allen TL, Flottemesch TJ, et al. An independent evaluation of four quantitative emergency department crowding scales. Acad Emerg Med. 2006; 13:1204–11.
24. McCarthy ML, Aronsky D, Jones ID, et al. The emergency department occupancy rate: a simple measure of emergency department crowding? Ann Emerg Med. 2008; 51:15–24.
25. West R. Objective standards for the emergency services: emergency admission to hospital. J R Soc Med. 2001; 94:4–8.
26. Wiler JL, Griffey RT, Olsen T. Review of modeling approaches for emergency department patient flow and crowding research. Acad Emerg Med. 2011; 18:1371–9.
27. Au L. Predicting overflow in an emergency department. IMA J Manag Mathemat. 2009; 20:39–49.
28. Abujudeh H, Vuong B, Baker SR. Quality and operations of portable X-ray examination procedures in the emergency room: queuing theory at work. Emerg Radiol. 2005; 11:262–6.
29. Cochran JK, Roche KT. A multi-class queuing network analysis methodology for improving hospital emergency department performance. Computers OR. 2009; 36:1497–512.
30. Huang X. A planning model for requirement of emergency beds. IMA J Math Appl Med Biol. 1995; 12:345–53.
31. Mayhew L, Smith D. Using queuing theory to analyse the government's 4-H completion time target in

accident and emergency departments. Health Care Manag Sci. 2008; 11:11–21.
32. Panayiotopoulos JC, Vassilacopoulos G. Simulating hospital emergency departments queuing systems: (GI/G/m(t)):(IHFF/N/∞). Eur J Operational Res. 1984; 18:250–8.
33. Siddharthan K, Jones WJ, Johnson JA. A priority queuing model to reduce waiting times in emergency care. Int J Health Care Qual Assur. 1996; 9:10–6.
34. Green LV, Soares J, Giglio JF, et al. Using queueing theory to increase the effectiveness of emergency department provider staffing. Acad Emerg Med. 2006; 13:61–8.
35. Stout WA, Tawney B. An Excel forecasting model to aid in decision making that effects hospital bed/resource utilization-hospital capability to admit emergency room patients. In: Bass EJ, editor. Proceedings of the 2005 Systems and Information Engineering Design Symposium; 2005. University of Virginia, April 29, 2005.
36. Broyles R, Cochran J. Estimating Business Loss to a Hospital Emergency Department from Patient Reneging by Queuing-Based Regression. IIE Industrial Engineering Annual Research Conference CD-ROM, Memphis, TN, 2007.
37. Roche KT, Cochran JK. Improving patient safety by maximizing fast-track benefits in the emergency department–a queuing network approach. IIE Industrial Engineering Research Conference. Witchita State University, Witchita, KS, 2007. CD-ROM, Memphis, TN, 2007.
38. Baker JR, Clayton ER, Taylor BW 3rd. A non-linear multi-criteria programming approach for determining county emergency medical service ambulance allocations. J Oper Res Soc. 1989; 40:423–32.
39. Scott DW, Factor LE, Gorry GA. Predicting the response time of an urban ambulance system. Health Serv Res. 1978; 13:404–17.
40. Barton GK. The wait for an ambulance. Queuing theory for managers. JEMS. 1986; 11:67–70.
41. Burwell TH, Mcknew MA, Jarvis JP. An application of a spatially distributed queuing model to an ambulance system. Socioecon Plann Sci. 1992; 26:289–300.
42. Whitt W. Engineering solution of a basic call-center model. Manag Sci. 2005; 51:221–35.
43. Brown L, Gans N, Mandelbaum A, et al. Statistical analysis of a telephone call center: a queueing-science perspective. J Am Stat Assoc. 2005; 100:36–50.
44. Allon SD, Deo S, Lin W. The impact of size and occupancy of hospitals on the extent of ambulance diversion: theory and evidence. Soc Sci Res Network. Kellogg School of Management, Northwestern University. Available at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1516843. Accessed Jul 9, 2013.
45. Cochran JK, Broyles JR. Developing nonlinear queuing regressions to increase emergency department patient safety: approximating reneging with balking. Comput Indust Engin. 2010; 59:378–86.
46. NIST/SEMATECH. E-handbook of Statistical Methods. 1.3.6.6.8. Weibull Distribution. Available at: http://www.itl.nist.gov/div898/handbook/eda/section3/eda3668.htm. Accessed Jul 11, 2013.
47. Eliason SR. Maximum Likelihood Estimation: Logic and Practice. Thousand Oaks, CA: Sage Publications, 1993.
48. McLachlan GJ, Krishan T. The EM Algorithm and Extensions. Hoboken, NJ: Wiley & Sons Inc., 2008.
49. Burt CW, McCaig LF, Valverde RH. Analysis of ambulance transports and diversions among U.S. emergency departments. Ann Emerg Med. 2005; 47:317–26.
50. Zohar E, Mandelbaum A, Shimkin N. Adaptive behavior of impatient customers in tele-queues: theory and empirical support. Manag Sci. 2002; 48:566–83.
51. Vonta F, Nikulin MS, Limnios N, Huber-Carol C, (editors). Sampling-Based Chi-Squared Test for Interval-Censored Data. Boston, MA: Birkhauser, 2008.

Supporting Information

The following supporting information is available in the online version of this paper:

Data Supplement S1. Description of the model derivation and specification.